Shear lift force on spherical bubbles

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The shear lift force on a spherical bubble in an unbounded shear flow at low Reynolds number is derived. It is two thirds of that for a solid sphere. An approximate expression for the shear lift force at finite Reynolds number and finite shear rate is obtained by an interpolation using the present result and Auton's result at large Re and small shear rate (Auton, T. R. 1987. *J. Fluid Mech.*, **183**, 190–218).

Keywords: shear lift; bubble; sphere

1. Introduction

The successful prediction of vapor bubble departure and lift-off from a heating surface in forced-convection boiling requires detailed knowledge of the hydrodynamic forces acting on the bubble (Klausner et al., 1992). An important force component contributing to the vapor bubble lift-off process, which can be important at moderate Reynolds number, is the shear lift force due to the mean flow gradient near the wall. Despite many efforts to predict both the inertial migration velocity and the shear lift force on a solid particle in bounded and unbounded shear flows (Saffman 1965; Ho and Leal 1974; Vasseur and Cox 1976; Cox and Hsu 1977; Drew 1978, 1988; Schonberg and Hinch 1989; McLaughlin 1991; Dandy and Dwyer 1990) and the shear lift force on a bubble in an inviscid shear flow (Auton 1987), little attention has been paid to the shear lift force on bubbles at low Reynolds number, let alone the case of nonspherical bubbles attached to a wall as found in forced-convection boiling. The objective of this work is to evaluate the shear lift force on a spherical bubble with negligible rotation in an unbounded shear flow in the low Reynolds number limit and to use an interpolation scheme to obtain a general approximation for the shear lift force at finite Reynolds number and shear rate. Such information is not currently available and warrants the attention it is given here.

Saffman (1965, 1968) derived the shear lift force on a solid sphere at zero Reynolds number to be

$$F_L = 6.46\rho\nu^{1/2}a^2(U - V)|dU/dy|^{1/2}\,\mathrm{sign}\,(dU/dy) \tag{1}$$

where ρ and ν are the fluid density and kinematic viscosity, respectively, a is the radius of the sphere, U and V are the respective velocities of the fluid and particle in the x-direction, and dU/dy is the shear rate of the mean flow. It is noted that the lift force is proportional to the square root of the shear rate. In deriving Equation 1, it was assumed that

$$\mathrm{Re}_s = v_S 2a/\nu \ll 1, \qquad (v_s = |U - V|) \tag{2}$$

$$\mathrm{Re}_G = G(2a)^2/\nu \ll 1, \qquad (G = |dU/dy|) \tag{3}$$

$$\mathrm{Re}_\Omega = \Omega(2a)^2/\nu \ll 1 \tag{4}$$

and

$$\mathrm{Re}_S \ll \mathrm{Re}_G^{1/2} \quad \text{or} \quad \varepsilon = \mathrm{Re}_G^{1/2}/\mathrm{Re}_S \gg 1 \tag{5}$$

where Ω is the rotational speed of the sphere. McLaughlin (1991) extended the result of Saffman to include arbitrary ε so that the restriction given by Equation 5 in Saffman's analysis is removed. The lift force for arbitrary ε can be expressed as

$$C_L/C_{LSa} = 0.443\,J(\varepsilon) \tag{6}$$

where

$$C_L = F_L/[\tfrac{1}{2}\pi a^2\rho v_s^2] \tag{7}$$

and the subscript "Sa" denotes Saffman's result, which is recovered as $J \to 2.255$ for large ε. McLaughlin found that $J(\varepsilon)$ decreases to zero rapidly as ε decreases. Numerical results for the shear lift force on a solid sphere were obtained by Dandy and Dwyer (1990) for Re_S ranging from 0.1 to 100 and for the dimensionless shear rate

$$\alpha = Ga/v_S = \tfrac{1}{2}\mathrm{Re}_S\varepsilon^2 \tag{8}$$

ranging from 0.005 to 0.4. Mei (1992) proposed a unified approximate expression for the shear lift force on a solid sphere for $\mathrm{Re}_S \le 100$ based on the results of Saffman (1965, 1968) and Dandy and Dwyer (1990).

Auton (1987) derived an expression for the shear lift force on a sphere in an inviscid shear flow, which is applicable to the motion of large spherical bubbles for $\alpha \ll 1$. It was found that the lift force in inviscid shear flow is proportional to α in the limits $\alpha \to 0$ and $\mathrm{Re} \to \infty$. The resulting lift coefficient is

$$C_L = \tfrac{3}{4}\alpha \tag{9}$$

In this work, uniform shear flow over a spherical bubble is considered. This differs from the problem considered by Saffman (1965) only in the boundary condition on the sphere surface. The shear stress-free condition is enforced on the bubble surface in lieu of the usual no-slip condition applied to the solid particle surface. To avoid the complex procedure of matching the inner and outer solutions, the approach outlined by McLaughlin (1991) is closely followed. By evaluating the inertial migration velocity, rather than the lift force directly, the shear lift force on the bubble can be easily obtained from the numerical results given by McLaughlin (1991) using the relationship between the lift force and the migration velocity given in Saffman (1965). An expression for the shear lift force at finite Reynolds number and finite shear is suggested by

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interpolating the present result at small Re and Auton's (1987) result at large Re and small shear. While the precise determination of the shear lift force on a nonspherical bubble attached to a wall is not yet possible, an understanding as to the effects of the shear rate and Reynolds number is gained, and qualitative predictions can be made. Furthermore, the result is directly applicable to vapor bubbles in forced-convection boiling that have already lifted off the heating surface as well as to small spherical bubbles in turbulent free shear flows.

2. Analysis

Consider a uniform shear flow over a stationary spherical bubble with velocity $\mathbf{v} = (Gx_1 + v_S)\mathbf{e}_3$ at infinity. Here $\mathbf{e}_3$ and x_3 are respectively the unit vector and coordinate in the direction of the undisturbed flow, and x_1 is the coordinate along which the undisturbed velocity varies with $\mathbf{x} = (x_1, x_2, x_3)$, constituting a Cartesian coordinate system. For $\mathrm{Re}_S \ll 1$, an inner region exists where the effects of inertia are small compared to the viscous effects and an outer region exists where the effects of inertia are comparable to the viscous effects (Proudman and Pearson 1957). A matched asymptotic method is therefore needed to systematically solve the flow field, thus including the effects of inertia in the lift force. Because the method we adopt here for the bubble is based on what has been successfully applied to a solid sphere, the solution procedure for evaluating the shear lift force over a solid sphere is briefly summarized as follows.

2.1 Shear flow over a solid sphere

For a shear flow over a solid sphere, Saffman (1965) showed that the calculation of the lift force is a singular perturbation problem and the effect of the sphere in the outer region can be replaced by a point force, $F_s = 6\pi\mu v_s a$, which is the Stokes drag at very small Reynolds number. The outer solutions for the velocity and the pressure were obtained in wavenumber space and expressed in the form of multidimensional integration. McLaughlin (1991) extended Saffman's work to arbitrary shear rate by retaining the additional convective term that arises from the shear. The resulting linearized momentum equation in the outer region was

$$(v_s + Gx_1)\frac{\partial \mathbf{v}}{\partial x_3} + Gv_1\mathbf{e}_3 = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} - \frac{\mathbf{F}}{\rho}\delta(\mathbf{x}) \qquad (10)$$

where p denotes pressure, $\mathbf{F}$ is the point force (which equals $6\pi\mu v_s a\mathbf{e}_3$ to the zeroth order in inertial effects), and $\delta(\mathbf{x})$ is the three-dimensional (3-D) delta function. Equation 10 in the outer field was then solved in wavenumber space. The shear lift force was evaluated numerically in terms of the inertial migration velocity, and hence tedious integration for the lift force based on the inner solution was avoided. McLaughlin's (1991) approach for evaluating the shear lift force on a solid particle is used here for the bubble.

2.2 Uniform flow over a sphere

For an unbounded liquid flow over a spherical bubble, the inner (creeping flow) solutions for the stream function ψ, vorticity ζ, and pressure p are

$$\psi(r, \theta) = \frac{v_s a^2}{2}[(r/a)^2 - r/a]\sin^2\theta \qquad (11)$$

$$\zeta(r, \theta) = -\frac{v_s a}{r^2}\sin\theta \qquad (12)$$

$$p(r, \theta) = -\mu\frac{v_s a}{r^2}\cos\theta \qquad (13)$$

(see Sherman 1990, for example), where r is the radius of position in the flow field measured from the center of the sphere and θ is measured from the rear stagnation point. The no-penetration and zero shear stress conditions are satisfied by Equation 11. The above solutions for $\zeta(r, \theta)$ and $p(r, \theta)$ are those for Stokes flow over a solid sphere reduced by a factor of two thirds. The vorticity and pressure given by Equations 12 and 13 are also that of a Stokeslet, $\zeta_{\mathrm{Stokeslet}}(r, \theta) = -(v_s a/r^2)\sin\theta$ and $p_{\mathrm{Stokeslet}}(r, \theta) = -\mu(v_s a/r^2)\cos\theta$. The stream function of the Stokeslet, $\psi_{\mathrm{Stokeslet}}(r, \theta) = -(rav_s/2)\sin^2\theta$, leads to the solution for a uniform flow over a bubble given by Equation 11 when it is combined with that of the uniform stream. The drag on the bubble in the flow direction is

$$F_b = 4\pi\mu v_s a \qquad (14)$$

which is two thirds of the Stokes drag on a solid sphere F_s. It is noted that the solutions $\zeta_{\mathrm{Stokeslet}}$ and $p_{\mathrm{Stokeslet}}$ are the only dynamically important contributions for $p(r, \theta)$ and $\zeta(r, \theta)$ as $r \to \infty$ in the Stokes region for a uniform flow over a bubble or solid sphere. Thus, when matching between the inner and the outer solution is rigorously pursued for a bubble using the approach outlined in Proudman and Pearson (1967), it can be readily shown that the outer solution for the stream function is exactly two thirds of that for a solid sphere to the first order in Re. This clearly demonstrates that the influence of the bubble (in the inner region) on the outer field can be approximated by a point force

$$\mathbf{F} = F_b\mathbf{e}_3 = 4\pi\mu v_s a\mathbf{e}_3 \qquad (15)$$

to the first order in the inertial effect. This observation is used, together with the method developed for the shear flow over a solid sphere, to derive an expression for shear lift force over a spherical bubble.

2.3 Shear flow over a bubble

The foregoing analyses suggest that momentum transport governing the outer field velocity in the case of shear flow over a bubble can also be described by Equation 10 with the point force $\mathbf{F}$ given by Equation 15. The rationale for the above approximation may also be explained from a physical point of view as follows. To an observer in the outer region, the sphere

Notation

a	Bubble radius (m)
C_L	Lift coefficient
$\mathbf{F}$	Force (N)
G	Magnitude of fluid velocity gradient (s^{-1})
Re	Reynolds number
$\mathbf{v}$	Fluid velocity in Cartesian coordinates (m/s)
v_s	Upstream fluid velocity seen by a bubble at its center (m/s)
$\mathbf{x}$	Cartesian coordinates (m)

Greek symbols

α	Dimensionless shear rate
ρ	Fluid density (kg/m^3)
ν	Kinetic viscosity (m^2/s)
μ	Dynamic viscosity (pa/s)

shrinks to a point as $r \to \infty$. Whether this sphere is immobile or free to move on its surface is immaterial as long as its effect on the outer field is correctly described, in this case through the point force given by Equation 15. An asymptotic analysis for the far field behavior of the Stokes solution for uniform flow over an arbitrary 3-D body is given in Batchelor (1967) and is also applicable to the case of uniform flow over a bubble.

Substituting $\mathbf{F} = F_b \mathbf{e}_3$ given by Equation 15 into Equation 10, following the detailed procedures outlined by McLaughlin (1991), the inertial migration velocity for the bubble due to the mean flow gradient can be obtained as

$$v_{bm} = \frac{1}{\pi^2} a v_s (G/\nu)^{1/2} J(\varepsilon), \text{ with } J(\infty) = 2.255 \tag{16}$$

which is two thirds of the migration velocity for a solid sphere found by McLaughlin. In the above, the subscript "b" denotes the result for a bubble. The numerical values of $J(\varepsilon)$, as well as some asymptotic expansions for $J(\varepsilon)$, were given by McLaughlin (1991). A curve fit based on the numerical values given by McLaughlin was obtained by Mei (1992) for $0.1 \le \varepsilon \le 20$ as

$$J(\varepsilon) \approx 0.6765\{1 + \tanh[2.5(\log_{10}\varepsilon + 0.191)]\} \times \{0.667 + \tanh[6(\varepsilon - 0.32)]\} \tag{17}$$

Thus, the shear lift force on a spherical bubble can be obtained as

$$F_{bL} = 1.91 J(\varepsilon) \rho \nu^{1/2} a^2 v_s |G|^{1/2} \operatorname{sign}(G) \tag{18}$$

In the limit $\varepsilon \to \infty$, F_{bL} is two thirds of the value for a solid sphere given by Saffman (1965).

3. Results and discussion

Equation 18 is a useful result because there are some generalizations that may be made to extend its applicability for engineering applications.

3.1 Extension to a fluid sphere of arbitrary viscosity

For a uniform creeping flow over a fluid sphere, the solutions for the stream function, vorticity, and pressure, as well as the hydrodynamic drag, have been obtained by Hadamard and Rybczynski (see Sherman 1990). The drag on the fluid sphere is

$$F_{fluid} = 6\pi\mu_o v_s a\,(3 + 2\mu_o/\mu_i)/(3 + 3\mu_o/\mu_i) \tag{19}$$

where μ_o and μ_i are the dynamic viscosities of the outer fluid and the fluid sphere, respectively.

Again, as the sphere shrinks to zero to an observer in the outer region, the boundary condition on the surface of the sphere is immaterial, and the effect of the sphere on the outer field can be approximated by a point force with $\mathbf{F} = F_{fluid}\mathbf{e}_3$. This leads to the following expression for the shear lift force on a fluid sphere

$$F_{fluid,L} = 2.865\, J(\varepsilon)(3 + 2\mu_o/\mu_i)/(3 + 3\mu_o/\mu_i) \times \rho \nu^{1/2} a^2 v_s |G|^{1/2} \operatorname{sign}(G) \tag{20}$$

which reduces to the result for a bubble as $\mu_o/\mu_i \to \infty$ and for a solid sphere as $\mu_o/\mu_i \to 0$.

3.2 An interpolation for finite Reynolds number

The result given by Equation 6 for the shear lift force on a solid sphere includes the effect of inertia at finite ε. The comparison between the results given by Equation 6 and the numerical results given by Dandy and Dwyer (1990) was not encouraging, as shown in Mei (1992), for small α or ε in the low Re limit. However, there are reasons to believe that the accuracy of the numerical results of Dandy and Dwyer (1990) at very small α or ε in the low Re limit is influenced by the relatively small size of the computational domain. As Dandy and Dwyer have reported, the sign of the computed lift force was different when the computational domain was varied from 10, 15, 20, and 25 radii of the sphere; also, their results were obtained using a computational domain of 25 radii. However, in the low Re limit, the lift force results from the inertia effect far away from the sphere, and it is likely that 25 radii are still too small a number to capture the inertial effect. Although there are no numerical results presently available for the shear force on a spherical bubble at small Re to verify the validity of Equation 18 for low and finite values of ε, the shear lift expression given by Equation 18 is perhaps the most reliable one for a bubble at low Re.

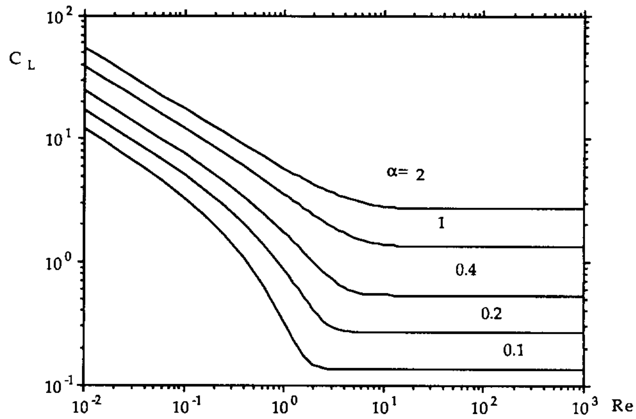

Figure 1. C_L as calculated from Equation 21, with the transition between the limits Re $\to 0$ and Re $\to \infty$ observed in the range Re $\sim$ 1–10

Based on the development of the interpolation formula given by Mei (1992) for the solid sphere, using the result of Auton (1987) for the large Re limit, assuming that the shear lift force on the bubble is linear in α in the large Re limit (as is the case for a solid sphere at Re ≥ 40), and using the present result (Equation 18) for the low Re limit, the following simple interpolation for the shear lift force has been obtained:

$$C_L = \alpha^{1/2}\left\{\left[\frac{1.72 J(\sqrt{2\alpha/\mathrm{Re}})}{\mathrm{Re}^{1/2}}\right]^n + \left(\frac{4}{3}\alpha^{1/2}\right)^n\right\}^{1/n}, \; n = 2 \tag{21}$$

with C_L defined by Equation 7. The above follows the analytical results only in the limits Re $\to 0$ or Re $\to \infty$ with small α. It is speculative in the intermediate ranges of Re and α. The suggested value for n ($=2$) is based on the expectation that the transition between these two limits occurs at Re $\approx$ 1–10, since in this range of Re the flow is neither viscous dominant nor inviscid. Figure 1 shows C_L based on the above interpolation. It is observed that the limiting values are recovered. Since, to the best knowledge of the authors, there is no expression yet available for the shear lift force on a spherical bubble for a large range of Reynolds number, the proposed interpolation (Equation 21) should be useful in estimating the shear lift force on a bubble in a shear flow.

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